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Single peak solitary wave solutions for the osmosis $K(2,2)$ equation under inhomogeneous boundary condition

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ABSTRACT

The qualitative theory of differential equations is applied to the $K(2,2)$ equation with osmosis dispersion. Smooth, peaked and cusped solitary wave solutions of the osmosis $K(2,2)$ equation under inhomogeneous boundary condition are obtained. The parametric conditions of existence of the smooth, peaked and cusped solitary wave solutions are given by using the phase portrait analytical technique. Asymptotic analysis and numerical simulations are provided for smooth, peaked and cusped solitary wave solutions of the osmosis $K(2,2)$ equation.

1. Introduction

It is well known that the study of nonlinear wave equations and their solutions are of great importance in many areas of physics. Travelling wave solution is an important type of solution for the nonlinear partial differential equation and many nonlinear partial differential equations have been found to have a variety of travelling wave solutions. For instance, the well-known Korteweg–de Vries equation

$$u_t - 6uu_x + uu_{xxx} = 0 \tag{1.1}$$

has smooth solitary wave solutions [1]. The Camassa–Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1.2}$$

was proposed by Camassa and Holm [2] as a model equation for unidirectional nonlinear dispersive waves in shallow water. This equation has attracted a lot of attention over the past decade due to its interesting mathematical properties. The Camassa–Holm equation has been found to has peakons, cuspons, stumpons and composite wave solutions [3].

In 1993, Rosenau and Hyman [4] introduced a genuinely nonlinear dispersive equation, a special type of KdV equation, of the form

$$u_t + a\left(u^m\right)_x + \left(u^n\right)_{xxx} = 0, \tag{1.3}$$

where a is a constant and both the convection term $(u^m)_x$ and the dispersion effect term $(u^n)_{xxx}$ are nonlinear. These equations arise in the process of understanding the role of nonlinear dispersion in the formation of structures like liquid

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drops. Rosenau and Hyman derived solutions called compactons for Eq. (1.3). Xu and Tian [5] introduced the osmosis $K(2,2)$ equation

$$u_t + \left(u^2\right)_x - \left(u^2\right)_{xxx} = 0, \tag{1.4}$$

where the negative coefficient of dispersion term denotes the contracting dispersion. They obtained the peaked solitary wave solution and the periodic cusp wave solution for Eq. (1.4). Zhou et al. [6] obtained two new types of travelling wave solutions called kink-like and antikink-like wave solutions. Zhou and Tian [7] obtained the analytic expressions of soliton solution of Eq. (1.4) by using the bifurcation method of dynamical systems. Zhou et al. [8] obtained expressions of the soliton and periodic wave solutions. Recently, Deng and Han [9] successfully found a peaked wave solution of Eq. (1.4) by using the first-integral method. More recently, Deng, Parkes and Cao [10] obtained some new exact travelling-wave solutions and stationary-wave solutions by using the auxiliary elliptic equation method.

In fact, it is important to consider the various boundary condition of travelling wave solutions. Qiao and Zhang [11] discussed the travelling wave solutions for the Camassa–Holm equation under the inhomogeneous boundary condition $\lim_{|x|\to\infty} u(x) = A$. Later, Zhang and Qiao [12] gave smooth and cusped soliton solutions of the Degasperis–Procesi equation

$$m_t - m_x u + 3mu_x = 0, \quad m = u - u_{xx}, \tag{1.5}$$

through setting the travelling wave mode under the boundary condition $\lim_{|x|\to\infty} u(x) = A$.

In this paper, we study the single peak solitary wave solutions of Eq. (1.4) under the inhomogeneous boundary condition

$$\lim_{x\to\pm\infty} u(x) = A. \tag{1.6}$$

The conditions of existence of the smooth, peaked and cusped solitary wave solutions are given by using the phase portrait analytical technique, which was developed by Li et al. [13–19]. We obtain all smooth, peaked and cusped solitary wave solutions of the osmosis $K(2,2)$ equation (1.4) and analyze their analytic and dynamical behavior.

The paper is organized as follows. In Section 2, we discuss the asymptotic behavior of solutions of the osmosis $K(2,2)$ equation (1.4). In Section 3, we give all single peak solitary wave solutions of the osmosis $K(2,2)$ equation (1.4) under the inhomogeneous boundary condition (1.6). A short conclusion is given in Section 4.

2. Asymptotic behavior of solutions

In this section, we first introduce some notations. Let $C^k(\Omega)$ denote the set of all k times continuously differential functions on the open set Ω. $L^p_{\mathrm{loc}}(R)$ refers to be the set of all functions whose restriction on any compact subset is L^p integrable. $H^1_{\mathrm{loc}}(R)$ stands for $H^1_{\mathrm{loc}}(R) = \{u \in L^2_{\mathrm{loc}}(R) \mid u' \in L^2_{\mathrm{loc}}(R)\}$.

Substituting $u(x,t) = u(\xi)$ and $\xi = x - ct$ into Eq. (1.4), we have

$$-cu_\xi + 2uu_\xi - 6u_\xi u_{\xi\xi} - 2uu_{\xi\xi\xi} = 0. \tag{2.1}$$

Integrating (2.1) once, we obtain

$$-cu + u^2 - 2(u_\xi)^2 - 2uu_{\xi\xi} = c_1, \tag{2.2}$$

where c_1 is an integration constant. Further, we get

$$(u_\xi)^2 = \frac{c_2}{u^2} + \frac{u^2}{4} - \frac{cu}{3} - \frac{c_1}{2}, \tag{2.3}$$

where c_2 is also an integration constant.

Let us solve (2.3) with the following inhomogeneous boundary condition

$$\lim_{\xi\to\pm\infty} u(\xi) = A, \tag{2.4}$$

where A is a constant. Eq. (2.3) can be cast into the following ordinary differential equation

$$(u_\xi)^2 = \frac{(u-A)^2(3u^2 + (6A-4c)u + 3A^2 - 2Ac)}{12u^2}. \tag{2.5}$$

The fact that both sides of Eq. (2.5) are nonnegative implies $3u^2 + (6A-4c)u + 3A^2 - 2Ac \geqslant 0$. If $4c^2 - 6Ac \geqslant 0$, then Eq. (2.5) reduces to

$$(u_\xi)^2 = \frac{(u-A)^2(u-B_1)(u-B_2)}{4u^2}, \tag{2.6}$$

where

$$B_1 = \frac{2c}{3} - A + \frac{\sqrt{4c^2 - 6Ac}}{3}, \qquad B_2 = \frac{2c}{3} - A - \frac{\sqrt{4c^2 - 6Ac}}{3}. \tag{2.7}$$

Obviously, $B_1 \geqslant B_2$. We assume that $c > 0$ throughout the paper, since there are similar results for $c < 0$.

Definition 2.1. A function $u(\xi)$ is said to be a single peak solitary wave solution for the osmosis $K(2,2)$ equation (1.4) if $u(\xi)$ satisfies the following conditions.

(A1) $u(\xi)$ is continuous on R and has a unique peak point ξ_0, where $u(\xi)$ attains its global maximum or minimum value.
(A2) $u(\xi) \in C^3(R - \{\xi_0\})$ satisfies (2.1) on $R - \{\xi_0\}$.
(A3) $\lim_{\xi \to \pm\infty} u(\xi) = A$.

Definition 2.2. A wave function u is called peakon if u is smooth locally on either side of ξ_0 and $\lim_{\xi \uparrow \xi_0} u_\xi(\xi) = -\lim_{\xi \downarrow \xi_0} u_\xi(\xi) = a$, $a \neq 0$, $a \neq \pm\infty$.

Definition 2.3. A wave function u is called cuspon if u is smooth locally on either side of ξ_0 and $\lim_{\xi \uparrow \xi_0} u_\xi(\xi) = -\lim_{\xi \downarrow \xi_0} u_\xi(\xi) = \pm\infty$.

Without losing the generality, we assume $\xi_0 = 0$.

Theorem 2.4. *Suppose that $u(\xi)$ is a single peak solitary wave solution for the osmosis $K(2,2)$ equation (1.4) at the peak point $\xi_0 = 0$. Then we have*

(i) *if $c < \frac{3}{2}A$, then $u(0) = 0$;*
(ii) *if $c \geqslant \frac{3}{2}A$, then $u(0) = 0$ or $u(0) = B_1$ or $u(0) = B_2$.*

Proof. If $u(0) \neq 0$, then $u(\xi) \neq 0$ for any $\xi \in R$ since $u(\xi) \in C^3(R - \{0\})$. Differentiating both sides of Eq. (2.5) yields $u(\xi) \in C^\infty(R)$.

(i) When $c < \frac{3}{2}A$, if $u(0) \neq 0$, then $u \in C^\infty(R)$. By the definition of single peak solution we have $u'(0) = 0$. However, by Eq. (2.5) we must have $u(0) = A$, which contradicts the fact that 0 is the unique peak point.

(ii) When $c \geqslant \frac{3}{2}A$, if $u(0) \neq 0$, by Eq. (2.5) we know $u'(0)$ exists. According to the definition of peak point, we have $u'(0) = 0$. Thus we obtain $u(0) = B_1$ or $u(0) = B_2$ from Eq. (2.6), since $u(0) = A$ contradicts the fact that 0 is the unique peak point. □

Theorem 2.5. *Suppose that $u(\xi)$ is a single peak solitary wave solution for the osmosis $K(2,2)$ equation (1.4) at the peak point $\xi_0 = 0$. Then we have the following solutions classification and asymptotic behavior.*

(i) *If $u(0) \neq 0$, then $u(\xi)$ is a smooth solitary wave solution.*
(ii) *If $u(0) = 0$ and $A = \frac{2}{3}c$, then $u(\xi)$ gives the peakon solution $\frac{2c}{3}(1 - e^{-\frac{1}{2}|x-ct|})$.*
(iii) *If $u(0) = 0$ and $A \neq \frac{2}{3}c$, then $u(\xi)$ is a cusped solitary wave solution and*

$$u(\xi) = \mu|\xi|^{\frac{1}{2}} + O\big(|\xi|\big), \quad \xi \to 0, \tag{2.8}$$

$$u'(\xi) = \frac{\mu}{2}|\xi|^{-\frac{1}{2}} \operatorname{sign}(\xi) + O(1), \quad \xi \to 0, \tag{2.9}$$

where $\mu = \pm\frac{1}{6}\sqrt{6|A|\sqrt{36A^2 - 24Ac}}$. Thus $u(\xi) \notin H^1_{\mathrm{loc}}(R)$.

Proof. (i) From the process of proving of Theorem 2.4, we know that if $u(0) \neq 0$, then $u(\xi)$ is a smooth solitary wave solution.

(ii) If $u(0) = 0$ and $A = \frac{2}{3}c$, then Eq. (2.5) becomes

$$(u_\xi)^2 = \frac{1}{4}\left(u - \frac{2}{3}c\right)^2. \tag{2.10}$$

Solving Eq. (2.10), we obtain the peakon solution

$$u(\xi) = \frac{2c}{3}\big(1 - e^{-\frac{1}{2}|x-ct|}\big). \tag{2.11}$$

(iii) If $u(0)=0$ and $A\neq\frac{2}{3}c$, then by the definition of single peak solitary wave solution we have $A\neq 0$, thus $3u^2+(6A-4c)u+3A^2-2Ac$ does not contain the factor u. From Eq. (2.5) we obtain

$$u_\xi=-\operatorname{sign}(A)\frac{u-A}{u}\sqrt{\frac{3u^2+(6A-4c)u+3A^2-2Ac}{12}}\operatorname{sign}(\xi). \tag{2.12}$$

Let $h(u)=\frac{\sqrt{12}}{(A-u)\sqrt{3u^2+(6A-4c)u+3A^2-2Ac}}$, then $h(0)=\frac{\sqrt{12}}{A\sqrt{3A^2-2Ac}}$ and

$$\int\operatorname{sign}(A)uh(u)\,du=\int\operatorname{sign}(\xi)\,d\xi. \tag{2.13}$$

Inserting $h(u)=h(0)+O(u)$ into Eq. (2.13) and using the initial condition $u(0)=0$, we obtain

$$\frac{1}{2}u^2\big|h(0)\big|\big(1+O(u)\big)=|\xi|. \tag{2.14}$$

Thus

$$u=\pm\sqrt{\frac{2}{|h(0)|}}|\xi|^{\frac{1}{2}}\big(1+O(u)\big)^{-\frac{1}{2}}, \tag{2.15}$$

which implies $u=O(|\xi|^2)$. Therefore we have

$$u(\xi)=\mu|\xi|^{\frac{1}{2}}+O\big(|\xi|\big),\quad \xi\to 0, \tag{2.16}$$

and

$$u'(\xi)=\frac{\mu}{2}|\xi|^{-\frac{1}{2}}\operatorname{sign}(\xi)+O(1),\quad \xi\to 0, \tag{2.17}$$

where $\mu=\pm\frac{1}{6}\sqrt{6|A|\sqrt{36A^2-24Ac}}$. Thus $u(\xi)\notin H^1_{\mathrm{loc}}(R)$. $\square$

3. Smooth, peaked and cusped single peak solitary wave solutions

Theorem 2.5 gives a classification for all single peak solitary wave solutions for the osmosis $K(2,2)$ equation (1.4). In this section, we will present all possible single peak solitary wave solutions and obtain some implicit solutions. We should discuss three cases: $c=\frac{3}{2}A$, $c>\frac{3}{2}A$ and $c<\frac{3}{2}A$.

Case I. $c=\frac{3}{2}A$.

If $c=\frac{3}{2}A$, then the only possible single peak solitary wave solution is the peakon solution

$$u(\xi)=A\big(1-e^{-\frac{1}{2}|x-ct|}\big). \tag{3.1}$$

The profile of peakon wave is shown in Fig. 2(2-1).

Case II. $c>\frac{3}{2}A$.

By virtue of Theorem 2.4 and Theorem 2.5 any single peak solitary wave solution for the osmosis $K(2,2)$ equation (1.4) must satisfy the following initial and boundary values problem

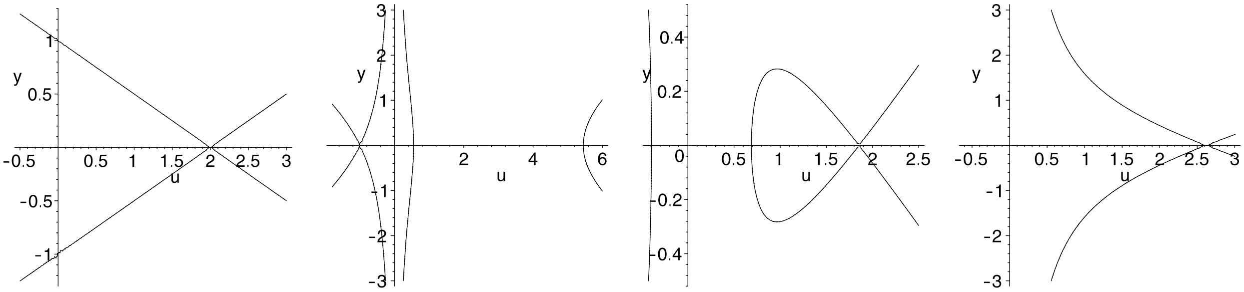
$$\begin{cases}(u_\xi)^2=\dfrac{(u-A)^2(u-B_1)(u-B_2)}{4u^2},\\ u(0)\in\{0,B_1,B_2\},\\ \lim\limits_{\xi\to\pm\infty}u(\xi)=A.\end{cases} \tag{3.2}$$

Eq. (3.2) implies

$$u\geqslant B_1\quad\text{or}\quad u\leqslant B_2, \tag{3.3}$$

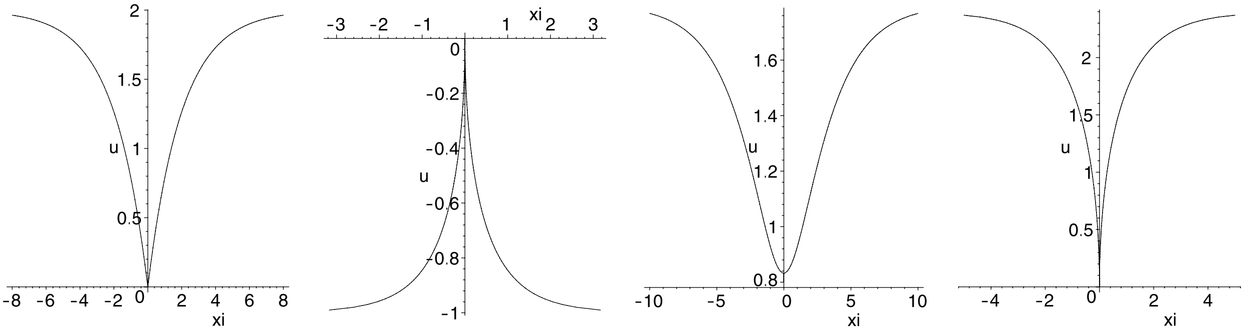
and

$$(A-B_1)(A-B_2)\geqslant 0. \tag{3.4}$$



(1–1) (1–2) (1–3) (1–4)

Fig. 1. Phase portraits of Eq. (2.5) on the (u, u_ξ) plane. (1-1) $2c - 3A = 0$. (1-2) $\alpha < 0$, $A < 0$. (1-3) $\alpha > 1$. (1-4) $\alpha < 0$, $A > 0$.



(2–1) (2–2) (2–3) (2–4)

Fig. 2. The profiles of waves. (2-1) Peakon, $A = 2$, $c = 3$. (2-2) Cuspon, $A = -1$, $c = 1$. (2-3) Smooth solitary wave, $A = 1.8$, $c = 3$. (2-4) Anti-cuspon, $A = 2.4$, $c = 2$.

Since $2c - 3A \neq 0$, introducing the constant $\alpha = \frac{A}{2c-3A}$ yield

$$\alpha(\alpha - 1) \geqslant 0, \tag{3.5}$$

which implies:

$$\alpha < 0; \qquad \alpha = 0; \qquad \alpha = 1; \qquad \alpha > 1. \tag{3.6}$$

Using the standard phase portrait analytical technique (see Fig. 1) and Theorem 2.4, we know that if $u(\xi)$ is a single peak solitary wave solution of the osmosis $K(2, 2)$ equation (1.4), then

$$u_\xi = -\frac{u - A}{2u}\sqrt{(u - B_1)(u - B_2)}\,\mathrm{sign}(\xi), \tag{3.7}$$

and

$$u(0) = \begin{cases} \max(0, B_1), & \text{if } u(0) \text{ is a minimum}, \\ \min(0, B_2), & \text{if } u(0) \text{ is a maximum}. \end{cases} \tag{3.8}$$

Taking the integration of both sides of Eq. (3.7) leads to

$$\int f(u)\,du = -|\xi|, \tag{3.9}$$

where $f(u) = \frac{2u}{(u-A)\sqrt{(u-B_1)(u-B_2)}}$. When $\alpha \neq 0, 1$, i.e. $4A^2 - 2Ac \neq 0$, we obtain

$$\int f(u)\,du = I_1(u) - \frac{2A}{\sqrt{4A^2 - 2Ac}} I_2(u) - K = \Phi(u) - K, \tag{3.10}$$

where

$$I_1(u) = \ln\left|\frac{B_1 + B_2}{2} - u - \sqrt{(u - B_1)(u - B_2)}\right|, \tag{3.11}$$

$$I_2(u) = \ln\left|\frac{(A-B_1)(u-B_2)+(A-B_2)(u-B_1)+2\sqrt{(A-B_1)(A-B_2)(u-B_1)(u-B_2)}}{u-A}\right|, \tag{3.12}$$

and K is an integration constant. Thus we obtain the implicit solution $u(\xi)$ defined by

$$\Phi(u) = -|\xi| + K. \tag{3.13}$$

Clearly,

$$I_1(B_1) = I_1(B_2) = \ln\left|\frac{B_1-B_2}{2}\right| = \ln\frac{\sqrt{4c^2-6Ac}}{3}, \tag{3.14}$$

$$I_2(B_1) = I_2(B_2) = \ln|B_1-B_2| = \ln\frac{2\sqrt{4c^2-6Ac}}{3}. \tag{3.15}$$

So, for $u(0) = B_1$ or B_2, the constant $K_0 = \Phi(u(0))$ is defined by

$$K_0 = \ln\frac{\sqrt{4c^2-6Ac}}{3} - \frac{2A}{\sqrt{4A^2-2Ac}}\ln\frac{2\sqrt{4c^2-6Ac}}{3}, \tag{3.16}$$

and for $u(0) = 0$,

$$K_0 = I_1(0) - \frac{2A}{\sqrt{4A^2-2Ac}} I_2(0). \tag{3.17}$$

(1) $\alpha < 0$.

If $\alpha < 0$, then

$$A < 0 < B_2 < B_1, \qquad u(0) = 0, \qquad A < u \leqslant 0.$$

From $f(u) < 0$, we know that $\Phi(u)$ is strictly decreasing on $(A, 0]$,

$$\Phi_1(u) = \Phi|_{(A,0]}(u), \tag{3.18}$$

gives a unique cusped solitary wave solution $u_1(\xi)$ satisfying the initial and boundary values condition (3.2) with $u_1(0) = 0$. Here, we give the following implicit expression of the cusped solitary wave solution,

$$I_3(u) - d_1 + \frac{A}{\sqrt{(A-B_1)(A-B_2)}}\big(I_4(u) - d_2\big) = -\frac{1}{2}|\xi|, \tag{3.19}$$

where

$$I_3(u) = -\ln\left|2u - B_1 - B_2 + 2\sqrt{(u-B_1)(u-B_2)}\right|, \tag{3.20}$$

$$d_1 = -\ln\left|2\sqrt{B_1B_2} - B_1 - B_2\right|, \tag{3.21}$$

$$I_4(u) = \ln\left|\frac{(A-B_1)(u-B_2)+(A-B_2)(u-B_1)+2\sqrt{(A-B_1)(A-B_2)(u-B_1)(u-B_2)}}{u-A}\right|, \tag{3.22}$$

$$d_2 = \ln\left|\frac{(A-B_1)B_2+(A-B_2)B_1-2\sqrt{(A-B_1)(A-B_2)B_1B_2}}{A}\right|. \tag{3.23}$$

The profile of cusped solitary wave solution is shown in Fig. 2(2-2).

(2) $\alpha = 0$.

In this case $A = 0$, Eq. (2.5) becomes

$$u_\xi = -\frac{1}{2}\sqrt{u\left(u-\frac{4c}{3}\right)}\operatorname{sign}(\xi), \quad u(\pm\infty) = 0. \tag{3.24}$$

Thus there is no single peak solitary wave solution for the above boundary condition.

(3) $\alpha = 1$.

If $\alpha = 1$, then $c = 2A, B_1 = A$ and $B_2 = -\frac{A}{3}$, there is no single peak solitary wave solution for the above boundary condition.

(4) $\alpha > 1$.

In this case, by the standard phase portrait analysis, we have

$$B_2 < 0 < B_1 < A, \qquad u(0) = B_1, \qquad B_1 \leqslant u < A.$$

From $f(u) < 0$, we know that $\Phi(u)$ is strictly decreasing on $[B_1, A)$,

$$\Phi_2(u) = \Phi|_{[B_1,A)}(u). \tag{3.25}$$

gives a smooth single peak solitary wave solution. Therefore $u_2(\xi) = \Phi_2^{-1}(-|\xi| + K_0)$ is the solution satisfying

$$u_2(0) = B_1, \qquad \lim_{\xi\to\pm\infty} u_2(\xi) = A, \qquad u_2'(0) = 0. \tag{3.26}$$

Further, we obtain the following implicit expression of the smooth solitary wave solution.

$$I_5(u) - \frac{A}{\sqrt{(A-B_1)(A-B_2)}} I_6(u) = \frac{1}{2}|\xi|, \tag{3.27}$$

where

$$I_5(u) = \ln\left|\frac{B_1 - B_2}{2u - B_1 - B_2 + 2\sqrt{(u-B_1)(u-B_2)}}\right|, \tag{3.28}$$

$$I_6(u) = \ln\left|\frac{(B_2 - B_1)(u - A)}{(A-B_1)(u-B_2) + (A-B_2)(u-B_1) + 2\sqrt{(A-B_1)(A-B_2)(u-B_1)(u-B_2)}}\right|. \tag{3.29}$$

The profile of smooth solitary wave solution is shown in Fig. 2(2-3).

Case III. $c < \frac{3}{2}A$.

In this case, according to Theorem 2.4 and standard phase portrait analytical technique, we have $u(0) = 0$, $0 \leqslant u < A$ and

$$u_\xi = -\frac{u-A}{u}\sqrt{\frac{3u^2 + (6A-4c)u + 3A^2 - 2Ac}{12}}\,\mathrm{sign}(\xi). \tag{3.30}$$

Let

$$X = u + a_1, \tag{3.31}$$

$$a_1 = \frac{6A - 4c}{6}, \tag{3.32}$$

$$a_2^2 = \frac{1}{9}\left(6Ac - 4c^2\right), \tag{3.33}$$

then Eq. (3.30) becomes

$$\frac{X - a_1}{(X - a_1 - A)\sqrt{X^2 + a_2^2}}\,dX = -\frac{1}{2}\,\mathrm{sign}(\xi)\,d\xi. \tag{3.34}$$

Integration of both sides of Eq. (3.34) gives

$$G(X) = -\frac{1}{2}|\xi| + K, \tag{3.35}$$

where

$$G(X) = J_1(X) - \frac{A}{\sqrt{(a_1+A)^2 + a_2^2}} J_2(X), \tag{3.36}$$

and

$$J_1(X) = \ln\left|X + \sqrt{X^2 + a_2^2}\right|, \tag{3.37}$$

$$J_2(X) = \ln\left|\frac{2(a_2^2 + (a_1+A)X + \sqrt{(a_1+A)^2 + a_2^2}\sqrt{X^2 + a_2^2})}{X - a_1 - A}\right|. \tag{3.38}$$

$G(X)$ is strictly decreasing on the interval $[a_1, a_1 + A)$. Define

$$G_1(X) = G|_{(a_1,a_1+A)}(X). \tag{3.39}$$

Then

$$G_1(X) = K_0 - \frac{1}{2}|\xi|, \tag{3.40}$$

where

$$K_0 = J_1(a_1) - \frac{A}{\sqrt{(a_1 + A)^2 + a_2^2}} J_2(a_1). \tag{3.41}$$

Since G_1 is a strictly decreasing function we can solve for X uniquely from Eq. (3.40) and obtain

$$u(\xi) = G_1^{-1}\left(K_0 - \frac{1}{2}|\xi|\right) - a_1, \tag{3.42}$$

which satisfies

$$u(0) = 0, \qquad \lim_{\xi\to\pm\infty} u(\xi) = A, \qquad u'(0+) = +\infty, \qquad u'(0-) = -\infty. \tag{3.43}$$

The profile of cusped solitary wave solution is shown in Fig. 2(2-4).

Let us summarize our results in the following theorem.

Theorem 3.1. *Suppose that $u(\xi)$ is a single peak solitary wave solution for the osmosis $K(2, 2)$ equation (1.4) at the peak point $\xi_0 = 0$, which satisfies the boundary condition (1.6). Then we have the following conclusions.*

(1) *If $2c - 3A = 0$, then u is the peakon solution*

$$u(x,t) = \frac{2c}{3}\left(1 - e^{-\frac{1}{2}|x-ct|}\right),$$

with the properties:

$$u(0) = 0, \qquad u(\pm\infty) = A, \qquad u'(0+) = \frac{c}{3}, \qquad u'(0-) = -\frac{c}{3}.$$

(2) *If $2c - 3A \neq 0$, let $\alpha = \frac{A}{2c-3A}$, then*

(i) *if $0 \leqslant \alpha \leqslant 1$, there is no solitary wave solution for the osmosis $K(2, 2)$ equation (1.4);*

(ii) *if $\alpha < 0$ and $A < 0$, the osmosis $K(2, 2)$ equation (1.4) has the cusped solitary wave solution*

$$u(x,t) = \Phi_1^{-1}\left(-|x - ct| + K_0\right),$$

with the properties:

$$u(0) = 0, \qquad u(\pm\infty) = A, \qquad u'(0+) = -\infty, \qquad u'(0-) = +\infty;$$

(iii) *if $\alpha > 1$, the osmosis $K(2, 2)$ equation (1.4) has the smooth solitary wave solution*

$$u(x,t) = \Phi_2^{-1}\left(-|x - ct| + K_0\right),$$

with the properties:

$$u(0) = B_1, \qquad u(\pm\infty) = A, \qquad u'(0) = 0;$$

(iv) *if $\alpha < 0$ and $A > 0$, the osmosis $K(2, 2)$ equation (1.4) has the cusped solitary wave solution*

$$u(x,t) = G_1^{-1}\left(-\frac{1}{2}|x - ct| + K_0\right) - a_1,$$

with the properties:

$$u(0) = 0, \qquad u(\pm\infty) = A, \qquad u'(0+) = +\infty, \qquad u'(0-) = -\infty.$$

4. Conclusion

In this paper, we study the single peak solitary wave solutions of the osmosis $K(2, 2)$ equation (1.4) under the inhomogeneous boundary condition. The conditions of existence of the smooth, peaked and cusped solitary wave solutions are given by using the phase portrait analytical technique. We obtain all smooth, peaked and cusped solitary wave solutions of the osmosis $K(2, 2)$ equation (1.4) and analyze their analytic and dynamical behavior.

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